September 2009

Extended uncertainty principle and the geometry of (anti)-de Sitter space

S. Mignemi[‡]

Dipartimento di Matematica, Università di Cagliari
viale Merello 92, 09123 Cagliari, Italy
and INFN, Sezione di Cagliari

Abstract

It has been proposed that on (anti)-de Sitter background, the Heisenberg uncertainty principle should be modified by the introduction of a term proportional to the cosmological constant. We show that this modification of the uncertainty principle can be derived straightforwardly from the geometric properties of (anti)-de Sitter spacetime. We also discuss the connection between the so-called extended generalized uncertainty principle and triply special relativity.

[‡] e-mail: smignemi@unica.it

1. Introduction

The Heisenberg uncertainty principle of quantum mechanics, $\Delta x_i \Delta p_j \geq \delta_{ij}/2$ plays a crucial role in the conceptual framework of quantum mechanics and can be considered one of cornerstones the of the theory. However, it may be possible that in extreme situations, far from the range of energy from which quantum mechanics was derived, the uncertainty principle needs to be modified.

For example, it has been argued that in an (anti)-de Sitter background the Heisenberg uncertainty principle should be modified by introducing corrections proportional to the cosmological constant $\Lambda = -3/l_H^2$, with l_H the (anti)-de Sitter radius[1], as [1]

$$\Delta x_i \Delta p_j \geq \frac{\delta_{ij}}{2} \left[1 + \frac{(\Delta x_i)^2}{l_H^2} \right]. \tag{1}$$

This modification has been called extended uncertainty principle (EUP) [2]. It has been motivated either by analogy with the generalized uncertainty principle (GUP), which is supposed to hold at quantum gravity scales, and postulates [3-5]

$$\Delta x_i \Delta p_j \geq \frac{\delta_{ij}}{2} \left[1 + l_P^2 (\Delta p_i)^2 \right], \tag{2}$$

with l_P the Planck length, or by gedanken experiments in which the expansion of the universe during a measurement is taken into account [6].

Although the correction to the Heisenberg formula given by (1) is negligible for cosmological values of l_H, it has a theoretical interest, since it can be used to derive the correct value of the temperature of an (anti)-de Sitter black hole[2] [1].

The original derivation of (1) was based essentially on analogy with the GUP. In this letter, we give a more cogent argument, showing that the EUP can be derived straightforwardly from the geometry of (anti)-de Sitter spacetime. In fact, in order to define quantum mechanics on a curved background, one must take into account its symmetries. In the case of (anti)-de Sitter spacetime, in particular, the generators of translations do not satisfy the same algebra as in flat space. It follows from the Jacobi identity that also the commutation relations between momentum and position coordinates are affected. Since the uncertainty principle follows from the relation

$$\Delta x_i \Delta p_j \geq \frac{1}{2} \left| < [x_i, p_j] > \right|, \tag{3}$$

where $< >$ denotes the expectation value , the nontriviality of the commutation relations implies its modification.

A straightforward extension of this argument also permits to derive the more general extended generalized uncertainty principle (EGUP), obtained by combining (1) and (2), from a model of doubly special relativity (DSR) on an (anti)-de Sitter background.

---

[1] In the following, in order to simplify the notations, we assume $l_H^2 < 0$ for de Sitter spacetime, and $l_H^2 > 0$ for (anti)-de Sitter . We use units such that $c = \hbar = 1$

[2] The Hawking temperature may also be obtained without introducing modifications of the uncertainty principle, but by using the dynamics of the gravitational field [7].

2. Generalized and extended uncertainty principles

The GUP (2) was first proposed in the context of string theories [3], and then derived for pure gravity from black hole gedanken experiments [4], non-commutative geometry [5], or models of measurements that included the effect of the dynamics of gravitational interactions [8].

For $l_P^2 > 0$, eq. (2) implies the existence of an absolute minimum in the position uncertainty, $x_{min} = 2l_P$. If one took instead $l_P^2 < 0$ no lower bound on measurable length would arise, but rather an upper limit on the momentum attainable by a particle, as in DSR theories [9]. These theories are based on a definition of spacetime in which the action of the Poincaré group is nonlinearly deformed in such a way that the Planck energy becomes an observer-independent constant, which sets an upper limit on the energy-momentum of elementary particles. Since such deformation is not unique several different models can be defined [10], starting from these assumptions.

In fact, a derivation of the GUP from a DSR model had been already proposed in ref. [5]. Another possibility is to obtain (2) from the Snyder model [11]. The Snyder model can be interpreted as an example of DSR defined on a noncommutative spacetime, in which Lorentz invariance is undeformed [12]. The commutation relations of the Snyder model read

$$[x_\mu, x_\nu] = i\, l_P^2\, J_{\mu\nu}, \qquad [x_\mu, p_\nu] = i\left(\eta_{\mu\nu} + l_P^2\, p_\mu p_\nu\right), \tag{4}$$

where $J_{\mu\nu}$ are the generators of Lorentz transformations and $\mu, \nu = 0, \ldots, 3$. The first equation displays the noncommutativity of the geometry. It is easy to see that the GUP follows from the second equation, when the expectation value of the right hand side is taken.

As remarked above, the usual DSR interpretation is obtained for $l_P^2 < 0$. When $l_P^2 > 0$, instead, the model has rather different physical properties, like the existence of a minimal length, that we shall discuss elsewhere [13]. For analogy with anti-de Sitter spacetime, we call the model with $l_P^2 > 0$ anti-Snyder.

The GUP can be used to obtain corrections to the heuristic derivation of the Hawking temperature for Schwarzschild black holes [14]. According to the standard argument, one may identify the uncertainty in the position of a particle emitted by Hawking effect as $\Delta x_i \approx r_+$, where r_+ is the radius of the horizon of the black hole, and its uncertainty in energy as $\Delta E \approx \Delta p_i \approx 1/(2\Delta x_i)$, where in the last step we have used Heisenberg uncertainty principle. Identifying ΔE with the Hawking temperature, one then obtains,

$$T \approx \frac{\Delta E}{2\pi} = \frac{1}{4\pi r_+}\,, \tag{5}$$

where the numerical factor $1/2\pi$ has been introduced in order to obtain the correct normalization.

If one had instead used the GUP, one should have used for Δp_i the value following from (2), and hence

$$T \approx \frac{\Delta E}{2\pi} = \frac{r_+}{8\pi l_P^2}\left[1 - \sqrt{1 - \frac{4l_P^2}{r_+^2}}\right], \tag{6}$$

which gives rise to corrections of order $(l_P/r_+)^2$ to the Hawking formula. If $l_P^2 > 0$, when the radius of the black hole reaches the minimal length $x_{min} = 2l_P$, the evaporation stops, leaving a remnant [14].

Duality in phase space suggested another generalization of the uncertainty principle [1], the EUP (1), that should hold on a curved background. In this case, the length scale l_H can been identified with the (anti)-de Sitter radius. In ref. [6] a better motivated derivation of the EUP was given, based on a gedanken experiment taking into account the effects due to the expansion of the universe during a measurement.

If $l_H^2 > 0$ (anti-de Sitter space), the relation (1) implies the existence of a minimal value for the momentum of a particle, given by $p_{min} = 2/l_H$, while if $l_H^2 < 0$ (de Sitter space) there is no constraint on the momentum, but a maximum length emerges, given of course by the radius l_H of the cosmological horizon. The existence of a minimum uncertainty on the momentum may be related to the existence of a lower bound for the mass of fields in anti-de Sitter space [15].

Also in the case of EUP one can obtain the temperature of a black hole following the same steps as above. It results

$$T \approx \frac{\Delta E}{2\pi} = \frac{1}{4\pi}\left[\frac{1}{r_+} + \frac{3r_+}{l_H^2}\right], \tag{7}$$

which is the standard temperature of an (anti)-de Sitter black hole [16].

3. Quantum mechanics in an (anti)-de Sitter background

One may suspect that the EUP can be derived from quantum mechanics defined on an (anti)-de Sitter background. The definition of quantum mechanics on (anti)-de Sitter spacetime poses nontrivial problems since, contrary to the case of flat spacetime, there is no privileged reference frame like the one singled out by the Minkowski metric, and in fact very little can be found on the subject in the literature [17]. However, it has been argued that a "natural" metric on (anti)-de Sitter spacetime can be defined by the choice of Beltrami (projective) coordinates [17-19]. In particular, geodesic motion takes place with constant velocity along straight lines in the spatial sections of this metric [18].

Here we do not attempt the nontrivial task of defining quantum mechanics on an (anti)-de Sitter background, but simply give a heuristic construction of the commutation relations of single-particle quantum mechanics in Beltrami coordinates, proceeding in analogy with the construction of the Poisson brackets of (anti)-de Sitter relativistic mechanics [19].

It is well known that (anti)-de Sitter spacetime can be realized as a hyperboloid of equation $\xi_A^2 = \pm\alpha^2$ embedded in 5-dimensional flat space, with coordinates ξ_A and metric tensor $\eta_{AB} = \mathrm{diag}\ (1, -1, -1, -1, \mp 1)$.

The isometries of (anti)-de Sitter spacetime are generated by the (anti)-de Sitter algebra. This can be identified with the Lorentz algebra $so(1, 4)$ (resp. $so(2, 3)$) of the 5-dimensional flat space, that leaves invariant the hyperboloid. Its interpretation as 4-dimensional (anti)-de Sitter algebra is obtained by splitting the generators into Lorentz

generators $J_{\mu\nu}$ and translation generators $p_\mu = J_{4\mu}/l_H$. The (anti)-de Sitter algebra can then be written as

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}\right),$$
$$[J_{\mu\nu}, p_\lambda] = i\left(\eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu\right), \qquad [p_\mu, p_\nu] = i\,\frac{J_{\mu\nu}}{l_H^2}\,. \tag{8}$$

It is natural to identify the generators of translations p_μ with the momentum operators of a quantum theory defined on (anti)-de Sitter spacetime. Of course, the position operators x_μ depend instead on the parametrization of the hyperboloid. It is important to remark that, due to the curvature of spacetime, the translation generators p_μ will have nontrivial commutation relations with the positions x_μ.

The Lorentz subalgebra of the 4-dimensional de Sitter algebra is identical to the flat space Lorentz algebra, and hence its generators have the usual commutation relations with the positions x_μ,

$$[J_{\mu\nu}, x_\lambda] = i\left(\eta_{\mu\lambda} x_\nu - \eta_{\nu\lambda} x_\mu\right), \tag{9}$$

but the commutation relations of the translation generators with the position variables depend instead on the specific choice of coordinates on the hyperboloid.

The most natural parametrization of the hyperboloid is given by projective (Beltrami) coordinates [17,18],

$$x_\mu = \frac{\xi_\mu}{\xi_4}. \tag{10}$$

With this choice [19],

$$[x_\mu, x_\nu] = 0, \qquad [x_\mu, p_\nu] = i\left(\eta_{\mu\nu} + \frac{x_\mu x_\nu}{l_H^2}\right). \tag{11}$$

From the second equation and (3), the EUP (1) follows straightforwardly.

Notice that the form of the commutation relations strongly depends on the parametrization of the hyperboloid. The coordinates usually adopted in general relativity, as cosmological or static coordinates, are not natural from a geometrical point of view, since they single out the time coordinate. In some sense, they resemble the Rindler coordinates on flat space. The realizations of quantum mechanics generated by these parametrizations are of course not equivalent to the one obtained here, and a discussion of their properties would require a deeper investigation of quantum mechanics on an (anti)-de Sitter background, that is beyond the scope of this letter.

4. Extended generalized uncertainty principle and triply special relativity

It is natural to combine (1) and (2) to obtain a more general uncertainty principle, which has been called EGUP [1,2],

$$\Delta x_i \Delta p_j \geq \delta_{ij}\left[1 + l_P^2 (\Delta p_i)^2 + \frac{(\Delta x_i)^2}{l_H^2}\right]. \tag{12}$$

In this case, if $l_P^2 > 0$ and $l_H^2 > 0$, a minimum measurable value both for length and for momentum results, and, as in the case of GUP, the evaporation process of a black hole stops when its radius reaches the minimal length.

The uncertainty principle (12) recalls the commutation relations of triply special relativity (TSR). TSR is a generalization of DSR to a curved background, based on a deformation of the (anti)-de Sitter algebra, such that both parameters l_P and l_H are observer independent. As in flat space DSR, choosing different deformations, one can define a variety of models exhibiting different properties [19,20].

In TSR models, the commutation relations of the (anti)-de Sitter group generators maintain the form (8), while the others commutation relations depend on the parametrization of the (anti)-de Sitter hyperboloid and on the specific deformation of the algebra. We consider here the choice of [20], which is a direct generalization of the choice of coordinates adopted in the previous section for (anti)-de Sitter spacetime, and of the Snyder deformation discussed in sect. 2. In this model, the commutation relations read

$$[x_\mu, x_\nu] = i\, l_P^2\, J_{\mu\nu}, \qquad [x_\mu, p_\nu] = i\left(\eta_{\mu\nu} + \frac{x_\mu x_\nu}{l_H^2} + l_P^2\, p_\mu p_\nu + 2\frac{l_P}{l_H}\, x_\mu p_\nu\right), \tag{13}$$

from which the GEUP (10) follows in view of (3).

As usual, depending on the choice of the signs of l_H^2 and l_P^2 one obtains different physical settings. For example, as mentioned above, in the anti-Snyder/anti-de Sitter case ($l_H^2 > 0$, $l_P^2 > 0$) a minimal value emerges both for length and momentum [2]. The temperature of a black hole is also modified and gives

$$T \approx \frac{\Delta E}{2\pi} = \frac{r_+}{8\pi l_P^2}\left[1 - \sqrt{1 - 4\frac{l_P^2}{r_+^2}\left(1 - \frac{r_+^2}{2l_H^2}\right)}\,\right]. \tag{14}$$

5. Conclusion

We have obtained the EUP from geometric considerations on (anti)-de Sitter spacetime. Our derivation has also been extended to the case of the EGUP by using a deformation of (anti)-de Sitter relativity, known as triply special relativity [19,20].

The temperature of black holes in (anti)-de Sitter background has sometimes been derived without making recourse to the modifications of the uncertainty principle [7]. In that case one obtains the corrections to the Schwarzschild temperature by introducing the gravitational interaction as an external force on a flat background, and neglecting the curvature of spacetime. One may interpret that approach as equivalent to ours, but based on semi-Newtonian gravity.

References

[1] B. Bolen and M. Cavaglià, Gen. Relativ. Grav. **37**, 1255 (2005).

[2] M.I. Park, Phys. Lett. **B659**, 698 (2008).

[3] G. Veneziano, Europhys. Lett. **2**, 199 (1986).

[4] M. Maggiore, Phys. Lett. **B304**, 63 (1993).

[5] M. Maggiore, Phys. Lett. **B319**, 83 (1993).

[6] C. Bambi and F.R. Urban, Class. Quantum Grav. **25**, 095006 (2008).

[7] F. Scardigli, gr-qc/0607010.

[8] F. Scardigli, Phys. Lett. **B452**, 39 (1999); R.J. Adler and D.I. Santiago, Mod. Phys. Lett. **A14**, 1371 (1999).

[9] G. Amelino-Camelia, Phys. Lett. **B510**, 255 (2001), Int. J. Mod. Phys. **D11**, 35 (2002).

[10] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. **243**, 90 (1995); J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002); F.J. Herranz, Phys. Lett. **B543**, 89 (2002); C. Heuson, gr-qc/0305015; G. Amelino-Camelia, Int. J. Mod. Phys. **D12**, 1211 (2003).

[11] H.S. Snyder, Phys. Rev. **71**, 38 (1947).

[12] S. Mignemi, Phys. Lett. **B672**, 186 (2009).

[13] S. Mignemi, in preparation.

[14] R.J. Adler, P. Chen and D.I. Santiago, Gen. Relativ. Grav. **33**, 2101 (2001); M. Cavaglià, S. Das and R. Maartens, Class. Quantum Grav. **20**, L205 (2003).

[15] P. Breitenlohner and D.Z. Freedman, Ann. Phys. **144**, 249 (1982).

[16] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15**, 10 (1977); S.W. Hawking and Page, Commun. Math. Phys. **87**, 577 (1983).

[17] R.L. Mallett and G.N. Fleming, J. Math. Phys. **14**, 45 (1973).

[18] H.Y. Guo, C.G. Huang, Z. Xu and B. Zhou, Phys. Lett. **A331**, 1 (2004); R. Aldrovandi R, J.P. Beltrán Almeida and J.P. Pereira, Class. Quantum Grav. **24**, 1385 (2007).

[19] S. Mignemi, arXiv:0802.1129; arXiv:0807.2186.

[20] J. Kowalski-Glikman and L. Smolin, Phys. Rev. **D70**, 065020 (2004).